

Source coding by efficient selection of ground states clusters

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In this letter, we show how the Survey Propagation algorithm can be generalized to include external forcing messages, and used to address selectively an exponential number of glassy ground states. These capabilities can be used to explore efficiently the space of solutions of random NP-complete constraint satisfaction problems, providing a direct experimental evidence of replica symmetry breaking in large-size instances. Finally, a new lossy data compression protocol is introduced, exploiting as a computational resource the clustered nature of the space of addressable states.

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The combinatorial problem of satisfying a large set of constraints that depend on N discrete variables is a fundamental one in computer science (optimization and coding theory) as well as in statistical physics (frustrated systems), and may become extraordinarily difficult even for randomly generated problems as soon as some control parameters are selected in specific ranges. While in general the study of the connection (if any) between worst-case and typical-case computational complexity of hard (NP-complete [1]) CSPs has not yet been fully developed, recent advances in the statistical mechanics study of random constraint satisfaction problems (CSPs) have connected the origin of such computational intractability to the onset of clustering in the space of optimal assignments and to the associated proliferation of metastable states [2, 3]. An important byproduct of the analytical studies of random CSPs has been the introduction of a new class of algorithms —the so called Survey Propagation (SP) algorithms [4, 5]— specially devised to deal with the clustering scenario and able to find optimal assignments of benchmark problems on which all known optimization algorithms fail.

In this letter we make a step forward in understanding the potentialities of Survey Propagation techniques by answering two questions:

- (i) Is it possible to explore efficiently the space of solutions of a given random combinatorial problem ?
- (ii) Can we use this capability of addressing a large set of states for computational purposes?

The first question relates to the physical issue of probing exactly the topology of the space of solutions (ground states) in problems that are in a clustered phase (the so called replica symmetry breaking (RSB) phase). Surprisingly enough, such geometrical insight is also important for engineering applications in information theory (e.g. LDPC error correcting codes [6]). The second question addresses a new algorithmic perspective in which the presence of many states becomes a powerful resource of the computational device.

In what follows we shall provide a positive answer to both questions by providing an efficient generalization of SP which is indeed capable of addressing efficiently

— computational cost almost linear in the size of the problems — an exponential number of different clusters of ground states that are invisible to the known search algorithms. From the physical side, we provide the first numerical evidence for the RSB geometric structure for large size instances of NP-complete problems. On the computational side we show that one can indeed take advantage of the addressability of the set of clusters of ground states to produce a “physical” lossy data compression scheme [7] with non-trivial performance.

A generic constraint satisfaction problem is defined by N discrete variables —e.g. Boolean variables, finite sets of colors or Ising spins— which interact through constraints involving typically a small number of variables. The energies $C_a(\vec{\xi})$ of the single constraints (equal to 0 or 1, depending if they are satisfied or not by a given assignment $\vec{\xi}$) sum up to give the global energy function \mathcal{E} of the problem and are function of just a small subset of variables $V(a) = \{j_{a_1}, \dots, j_{a_K}\}$ (every variable j is involved on the other hand in a subset $V(j)$ of constraints). Since one is interested in satisfying all the constraints simultaneously, the CSP is just equivalent to the problem of looking for zero energy ground states. For most NP-complete CSP the function \mathcal{E} can be directly interpreted as a spin-glass-like hamiltonian [3, 4, 8]. For instance, the well known case of the random K-SAT problem consists in deciding if M clauses —taking the form of the OR function of K variables chosen randomly among N possible ones— can be simultaneously true. The energy contribution associated to a single clause can then be written as $C_a(\vec{\xi}) = \prod_{l=1}^K \frac{1}{2}(1 + J_{a,l}x_{a_l})$, where $x_{a_l} = \pm 1$ depending on the truth value of j_{a_l} and $J_{a,l} = \pm 1$ if j_{a_l} appears negated or directed in the clause a . The same variable can appear directed and negated in different terms and hence give rise potentially to frustration.

The connectivity distribution and the loop structure of the *factor graph* associated to a CSP (in Fig. 1, variables are represented as circles connected to the constraints in which they are involved, depicted as squares) has a strong influence on the behavior of search algorithms. For many important random CSPs, when the ratio $\alpha = \frac{M}{N}$ is included in a narrow region $\alpha_d < \alpha < \alpha_c$ —the exact values

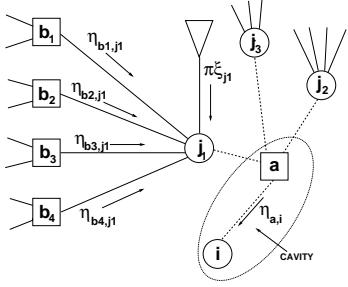


FIG. 1: A portion of factor graph

of the thresholds depending on the details of the problem and on the chosen random graph ensemble — the problem is still satisfiable but the zero energy phase of the associated hamiltonian breaks down in an exponential number of clustered components.

The cavity method of statistical physics, used for accurate analytical computations [4, 9, 10] of the threshold locations in good agreement with the numerical experiment [11], provides as well the theoretical foundation of the Survey Propagation (SP) message passing algorithm, successful in the resolution of instances of both the q -coloring and the K -SAT problem [4, 8] which are hard for local search algorithms. A constraint node b is supposed to send a message $\vec{u}_{b \rightarrow j} = \vec{e}_s$ (where \vec{e}_s are vectors having just the s -th component equal to 1) to a variable j each time that j would violate the constraint b assuming the value s (a set of messages corresponds then to a cluster of configurations). For locally tree-like factor graphs (like the ones associated typically to randomly generated instances) the messages incoming to $j \in V(a) \setminus i$ can be assumed uncorrelated, after the temporary removal of a single clause a and of a variable i (the so called cavity step, see Fig. 1). It becomes then possible to evaluate probability density functions for the messages, called *cavity surveys* (the probability space originates from the set of all clusters of satisfying assignments sampled with uniform measure):

$$Q_{b,j}(\vec{u}_{b,j}) = \eta_{b,j}^0 \delta(\vec{u}_{b,j}, \vec{0}) + \sum_{s=1}^q \eta_{b,j}^s \delta(\vec{u}_{b,j}, \vec{e}_s) \quad (1)$$

Here, $\eta_{b,j}^s$ are the probabilities that b constrains j not to enter the s state and $\eta_{b,j}^0$ is the probability that no message is sent, b being already satisfied by the assignment of other variables.

The cavity surveys form a closed system of KM functional equations for which a solution can be found in linear time by iteration [4, 5]:

$$Q_{a,i}(\vec{u}_{a,i}) = \int \mathcal{D}Q \delta_{\mathcal{E}}(\{\vec{u}_{b,j}\}) \chi(\vec{u}_{a,i}, \{\vec{u}_{b,j}\}) \quad (2)$$

where the function $\chi(\{\vec{u}_{b,j}\})$ depends on the specific CSP and $\mathcal{D}Q = \prod_{j \in V(a) \setminus i} \prod_{b \in V(j) \setminus a} Q_{b,j}(\vec{u}_{b,j})$. The functional $\delta_{\mathcal{E}}(\{\vec{u}_{b,j}\})$ acts as a filter, assigning null weight

to sets of messages associated to clusters of excited configurations. At the fixed point, from the knowledge of the surveys one may compute the fractions W_j^s (W_j^0) of clusters of solutions in which a variable j is frozen in the direction s (or is unfrozen). Such microscopic information can be successfully used to find optimal assignments by decimation [5].

We shall now present a generalization of the SP algorithm (SP-ext), allowing the retrieval of a solution close to any desired configuration $\vec{\xi}$. Hereafter we shall refer for simplicity to the K -SAT problem ($s = \pm 1$ only), but the method could be easily extended to a generic CSP. On general grounds, a way to analyze specific regions of the configuration space would be to solve the cavity equations in presence of an additional field conjugated to some geometrical constraint (e.g. fixed magnetization). This strategy would be however algorithmically inefficient in that it would change the nature of the components of the messages (from integers to reals), slowing down significantly the iterative solution of the SP equations.

Here we consider instead an arbitrary but quite natural extension of the SP equations in which external messages $\vec{u}_i = \vec{e}_{\xi_i}$ (represented in Fig. 1 as triangles) in an arbitrary direction $\xi \in \{-1, 1\}^N$ are introduced for each variable. New associated surveys $Q_i^\pi(\vec{u}_i) = (1 - \pi)\delta(\vec{u}_i, \vec{0}) + \pi\delta(\vec{u}_i, \vec{e}_{\xi_i})$ are given *a priori* and never updated, and affect dynamically the relative weight of the different clusters, entering into the measure $\mathcal{D}Q$ in the convolution integrals (2). The parameter π can be interpreted as strength of the perturbation. Convergence can be reached only if the zero energy constraint is respected. While an intensity $\pi \simeq 1$ would produce a complete polarization of the messages if $\vec{\xi}$ was a solution, in the general case the use of a smaller forcing intensity allows the system to react to the contradictory driving and to converge to a set of surveys sufficiently biased in the desired direction, allowing for an efficient selective exploration of specific parts of the solution space.

In order to use SP-ext for probing the local geometry of the zero energy phase one proceeds as follows. First, a random solution $\vec{\sigma}$ is found by decimation (SP-ext, differently from the standard SP [5], is typically able to retrieve complete solutions). Next, new satisfying assignments $\vec{\sigma}_\delta$ are generated, forcing now the system along a direction obtained flipping $N\delta$ spins of the original solution σ . In order to have a highly homogeneous distribution of the clusters, we have chosen for our experiments an ensemble of random K-SAT in which variables have fixed degree and are balanced (i.e. have an equal number of directed and negated occurrences in the clauses) For this specific ensemble and for $K = 5$, one has $\alpha_d = 14.8$ and $\alpha_c = 19.53$. Between α_d and $\alpha_G = 16.77$ the phase is expected to have multiple levels of clustering (a full RSB phase [9, 10]) whereas between α_G and α_c the 1-RSB phase is stable. We have estimated α_G with a new

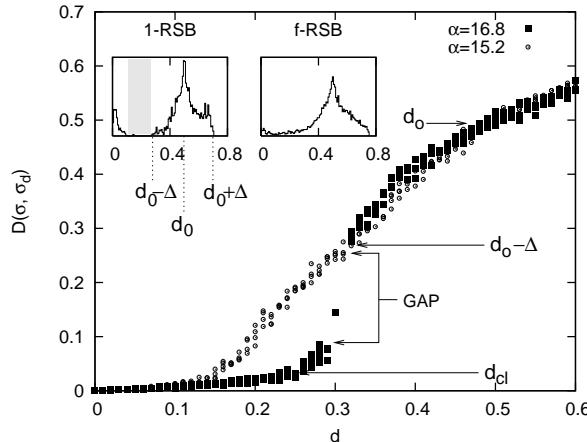


FIG. 2: Distribution of distances

message passing algorithm implementing the cavity equations at the level of two steps of RSB [12].

In the experiments we have taken instances of size $N = 10^4$ with an intensity $\pi = .35$ of the forcing (close to the highest value of π for which the SP-ext equations always converged for this sample). The Hamming distance between σ and σ_δ is plotted against δ in the first *stability diagram* (black data points) shown in Fig. 2. For $\delta < \delta_c \simeq 0.3$, $d_H(\vec{\sigma}, \vec{\zeta}_\delta)$ linearly increases with a very small slope until a value d_{cl} . Conversely, for $\delta > \delta_c$, it jumps to a value $d_0 - \Delta$, and a symmetric distribution of distances around d_0 is obtained. Under the hypothesis of homogeneous distribution of clusters, the fixed point average site magnetization $\langle W^+ - W^- \rangle$ provides an analytic estimation of the typical overlap $q_0 = \frac{1-d_0}{2}$ between two different clusters in agreement with the experiments. On the other hand, d_{cl} is of the order of the average fraction of unfrozen variables $\langle W^0 \rangle$. The gap between clusters is the main prediction of the 1-RSB cavity theory which finds in these experiments a nice confirm.

A completely different behavior is observed when repeating the experiment in the expected full RSB phase. The histogram of the reciprocal distances among all the generated solutions (inset of Fig. 2, gray bars) is now gapless, unlike the previous sample (black bars), and the related stability plot in the main figure (white data-points) deviates significantly from the 1-RSB case. Various shapes of the stability plot, differing for their degree of convexity, can be obtained starting from different solutions of the same sample. This hints to the existence of a mixed phase, in which higher order hierarchical clustering is present and in which many local cluster distribution topologies can coexist.

The ability of SP-ext to select clusters is a new computational feature which may play a role in different systems in which clustering of input data is important. Here we consider a first basic applications by implementing a lossy data compressor [7] which exploits the 1-RSB clustered

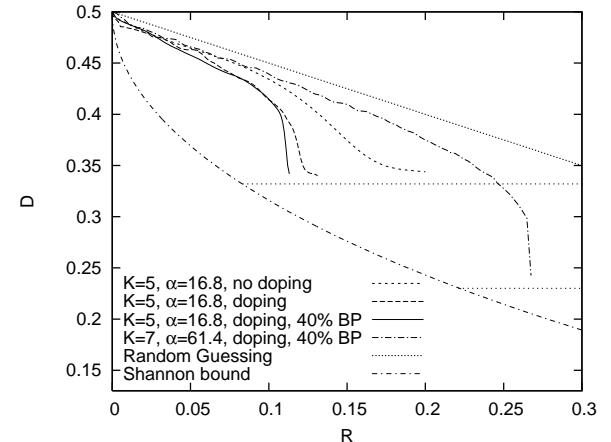


FIG. 3: Rate-Distortion profile for the compression of a random unbiased source

structure for data quantization purposes.

Let us suppose to have an input N -bit binary string $\vec{\xi}$ generated from an unbiased and uncorrelated random source. Given an appropriate K -SAT instance with N variables, a solution $\vec{\sigma}_\xi$ as close as possible to $\vec{\xi}$ can be generated with SP-ext. One can expect to find a solution at a distance close to $d_0 - \Delta$, if the cluster distribution is homogeneous (balanced and fixed even connectivity instances are then chosen). Furthermore, α is taken slightly larger than α_G , in order to maximize the number of addressable clusters, still preserving a sharp separation among them. At this point, a compressed string $\vec{\sigma}_\xi^R$ is built by retaining just the spins of the first NR variables of $\vec{\sigma}_\xi$. In the decompression stage, SP-ext is run over the same graph, applying a very intense forcing ($\pi = 0.99$) parallel to $\vec{\sigma}_\xi^R$. If $R > R_c$, SP-ext becomes able to exactly select the single cluster to which $\vec{\sigma}_\xi$ belongs. The cluster addressing is actually so sharp, that no decimation is needed and all the remaining $N(1 - R)$ unforced variables can simultaneously be fixed to their preferred orientation without creating contradictions. A comparison with the theoretical Shannon Bound [7] is done in Fig. 3 (dotted line), where the cluster selection transition is clearly visible. The accumulated distortion with respect to $\vec{\xi}$ is of the order of $d_{cl} + d_0 - \Delta$ (the horizontal lines in Fig. 3 refer to the original distance between $\vec{\xi}$ and $\vec{\sigma}_\xi$, and, obviously, no better distortion can be achieved in this scheme). The line relative to the performance of a trivial decoder in which the missing bits are randomly guessed is also plotted.

The Shannon Bound can be approached by the use of *iterative doping* [13] technique for choosing the bits to store. After the determination of $\vec{\sigma}_\xi$, SP-ext is run again *without* applying any forcing and a ranking of the most balanced variables is performed. One looks for the variable i which minimizes $|W_i^+ - W_i^-| + W_i^0$ (frozen in opposite directions in a similar number of clusters and rarely

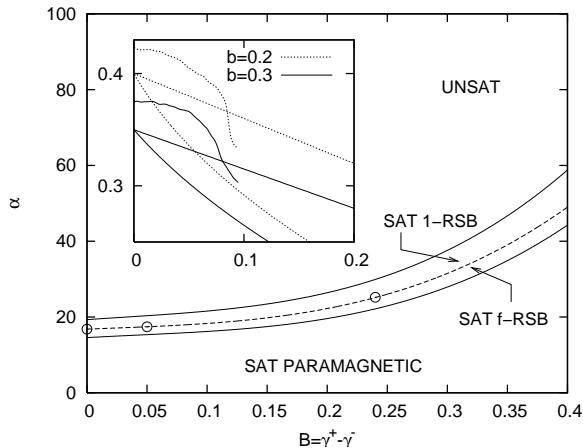


FIG. 4: Phase diagram for fixed degree biased 5-SAT and compression performance for two biased sources.

unconstrained). The state assumed by i in the solution $\vec{\sigma}_\xi$ will be taken as the first bit of the compressed string $\vec{\sigma}_\xi^R$ and used to fix i . New doping steps are done, until when the desired compression rate has been reached. An *identical* doping stage is then performed in decompression. The iterative ranking allows indeed to find out which spins have to be fixed accordingly to the ordered bit sequence $\vec{\sigma}_\xi^R$ and the left variables can be fixed as in the previous decompression method. Fixing a balanced variable “switches off” a larger number of clusters, and, hence, the critical rate is reduced (dash dotted curve in Fig. 3), thanks to a less redundant coding of the information needed for the cluster selection. A further improvement can be obtained, by using in the doping stage a modified iteration, that interpolates between SP and the well known Belief Propagation [14] equations (solid curve in Fig. 3).

SP-ext can also be applied when the occurrence probability P_\pm of the possible input symbols are different. The issue is of practical relevance since correlated sources can often be shown to be equivalent to memoryless biased sources [15]. Let suppose that the bias of the source is $b = P_+ - P_- > 0$. It is possible to engineer graphs with a cluster distribution concentrated around the ferromagnetic direction, by making for every variable i the fraction γ_+ of couplings $J_{a,i} = +1$ larger than the fraction γ_- of $J_{a,i} = -1$. As shown indeed in Fig. 4, a narrow SAT RSB stripe is still present for a balancing $B = \gamma_+ - \gamma_- < 0.435$. When the balancing is too large, there is on the other hand a direct transition between an unfrozen SAT phase and the UNSAT region (the line where optimized heuristics start to fail in determining a solution approximately continues the 1-RSB SAT/UNSAT transition line). The rate-distortion profile of the compression of a random uncorrelated source with $b = 0.2$ and $b = 0.3$ is shown in the inset of the same figure. The best found graphs for all the analyzed values of b are always located in prox-

imity of the Gardner line (empty circles in Fig. 4, better critical rates but worse distortions are found for higher connectivities). Cluster selection is still possible and ensures again a non-random correlation between the input and the output string. It is expected nevertheless that much better performances can be achieved by a careful optimization of the graph ensemble as it was shown for iterative decoding with Belief Propagation [16].

We conclude by noticing that work is in progress in order to obtain a fully local version of the decimation procedure leading to the different solutions, by mean of a local reinforcement technique: this fact might be important for the parallelization of the SP algorithm and for modeling distributed computation in complex networks.

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